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COMMUTATIVE MATRICES OF CLASS 4A AND ITS ABSTRACT CHARACTERISTICS

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ABSTRACT

The contents of this paper is an extension to a new class, class 4-A to our previous work on square matrices of Class – 3.These two classes differ by their general form and algebraic nature. An attempt has been made to search for and finally establish a class of (infinite set) of square matrices where in commutative property for matrix multiplication is preserved. In addition to this, Eigen values and Eigen vectors of such commutative matrices have dominant role to understand salient features of the class under discussion. It is, as shown, the most dominating property that libra values are preserved under matrix addition, multiplication, and matrix inversion operation.

KEYWORDS: Classes, Libra Value, Commutative Property, Eigen Value

NOTATION: Class 4 – A, CJ4 – A ((n x n, L(A) = P), ZL*, CJ4 – A((n x n, L(A) = 3P + K), P4**

- ZL * – A class of square matrices for which Libra Value is Zero.
- P4** -- A property associated with the sum of all the entries of any row and any column and sum of all the entries of Non- leading diagonal.
- P and K are real values.

1. INTRODUCTION

It has always remained our continued efforts in quest of an infinite class of matrices such that any two member matrices exhibit commutative property for matrix multiplication and finally we settle down in agreement to commutatively. We have classified square matrices in 5 broad classes. Each class is an infinite one with a dominating property either in sum of all the elements in any row or any column and both. Again to this we add a constraint that involves all the entries of either leading diagonal or non-leading diagonal or both. Classes are in correspondence to the property.

We have introduced an infinite class; we call it class 4 of square matrices which satisfy property P4, it possesses the property that the algebraic sum of each of its row, each of its column, and all entries of non-leading diagonal elements remains a fixed real constant. This constant, as mentioned in above cases, is called the Libra value of the matrix.

These matrices show remarkable properties in connection to algebraic structure of matrices. As a part of this class CJ4-A (n x n, L (A) = p), The members of the class CCJ4-A follow commutative property for matrix multiplication. It is this class which gets wide open, entries to the infinite classes observing commutative property for multiplication. In context to Eigen values and Eigen vectors of the matrices of this class, we have universality for all the member matrices. Also, their inverses do not deviate far away from this property and show fair closeness to this nature. Symmetric matrices as a sub –class to this set, add much to the existing properties.

On introducing these two infinite classes of square matrices, we now focus on class 4-A.

An n x n square matrix is said to be of the class 4A if it possesses the property that the algebraic sum of each of its row, each of its column, and all entries of non-leading diagonal elements remains a fixed real constant. This constant, as mentioned in above cases, is called the Libra value of the matrix.

One easier form of order n x n matrix of class 4A we shall follow in the notes to follow is,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot\cdot & a_{1\,n-1} & p-\left(\sum a_{1\,j}\right) \\ a_{21} & a_{22} & \cdot\cdot & a_{2\,n-1} & p-\left(\sum a_{2\,j}\right) \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ a_{n-1\,1} & a_{n-1\,2} & \cdot\cdot & a_{n-1\,n-1} & p-\left(\sum a_{n-1\,j}\right) \\ p-\left(\sum a_{i\,1}\right) & p-\left(\sum a_{i2}\right) & \cdot\cdot p-\left(\sum a_{in-1}\right) & & -(n-1)p+\sum\sum a_{i\,j} \end{bmatrix} \quad \textbf{(1)}$$

All entries are real values. The subscripts 'i and j' run over summation notation from 1 to (n-1)

The same real constant, found on adding either all the entries of any row or any column or all the entries of non-leading diagonal, is known as Libra value and it is denoted as L(A) = p; p ∈ R

There are n row equations; and n column equations, and one equation showing the constant sum of non-leading diagonal entries. Number of linearly independent equations are [n + n +1] -1 = 2n.

From n x n = n^2 entries in the given matrix of order n x n, 2n entries are basic variables and this implies that we have a free choice of n^2 – 2n = n(n-2) entries. An important point is that from **n** entries of non-leading diagonal, except the first and the last entry any other entry cannot be a free choice. This logic helps construct a n x n square matrix of class 4-A with above constraints.

The matrix given by (1) above represents a general form of matrices of class 4A; we denote it symbolically as follows.

A ∈ $CJ4A$ (n x n, L (A) = p)

We consider particular cases for n = 3 and 4.

For n= 3, there is a choice of 3 arbitrary elements denoted here as **a, b, and c.**

$$A = \begin{pmatrix} \boldsymbol{a} & b & p-(a+b) \\ c & -p+(2a+b+c) & 2p-(2a+2c+b) \\ p-(a+c) & 2p-(2a+2b-c) & -p+(3a+2b+2c) \end{pmatrix} \in CJ4A(\text{n x n, L (A) = p}) \quad \textbf{(2)}$$

Three elements viz. **a, b,** and **c** are arbitrary real values while the other elements are controlled by the Libra value constraint (L (A) = p)

E.G. $A=\begin{bmatrix} 2 & 3 & -1 \\ 5 & 8 & -9 \\ -3 & -7 & 14 \end{bmatrix} \in CJ4A$(3x3, L (A) = 4)

For n= 4, there is a choice of 8 [= n^2 – 2n for n = 4] arbitrary elements.

$A=\begin{bmatrix} 3 & -2 & 5 & 3 \\ 2 & 3 & -3 & 7 \\ -4 & 1 & 0 & 12 \\ 8 & 7 & 7 & -13 \end{bmatrix} \in CJ4A$(4x4, L (A) = 9)

1.1 Special Cases, Determinant, and the class Z_L

In this section we discuss special form of matrices of class-4A and identify algebraic property. As a special case to the libra value L (A) = p = 0, we discuss a special class Z_L.

(a) General Form and Determinant Value

We consider the general form square matrices of class 4A of order 3 x3.

$$A = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} & p-(a+b) \\ \boldsymbol{c} & -p+(2a+b+c) & 2p-(2a+2c+b) \\ p-(a+c) & 2p-(2a+2b-c) & -p+(3a+2b+2c) \end{pmatrix} \in CJ4A(\text{n x n, L (A) = p}) \quad \textbf{(2)}$$

Where **a, b, c,** and p are real values. We mention here some important properties.

Det. A=|A| = p [(p-3a)^2 + 3(a-b)(c-a)] **(3)**

Det. |A|= |A|= 0 has one or more cases

(1) **p = 0** (2) **p=3a and a = b (3)p = 3a and a=c**

In the case (2) and (3) the matrix **A** will have linearly dependent row vectors forcing |**A**|= 0

(b) Zero Libra Class Z_L

Matrices of class 4A for which |A|= 0, i.e. singular matrices, form a sub-class of class 4-A.

This class is denoted as **Z_L. In fact Z_L** = $CJ4A$(3 x 3, L(A) = p=0) ⊂ $CJ4A$(3 x 3, L(A) = p); p ∈R

For p = 0, we have the general form of class 4-A matrix as follows.

$$A = \begin{bmatrix} a & b & -(a+b) \\ c & (2a+b+c) & -(2a+b+2c) \\ -(a+c) & -(2a+2b+c) & (3a+2b+2c) \end{bmatrix} \in Z_L \subset CJ4A(\text{3 x 3, L(A) = p = 0}) \quad \textbf{(4)}$$

and clearly |A|= 0

In the above example, we take a= 3, b = -4, and c = 1

E.G. $A=\begin{bmatrix} 3 & -4 & 1 \\ 1 & 3 & -4 \\ -4 & 1 & 3 \end{bmatrix} \in Z_L \subset CJ4A$(3 x 3, L(A) = p = 0) and hence |A| = 0

(c) Identity Matrix and Null Matrix:

We consider the general form of class 4A matrices given by the relation (2)

For **a = 1, b = 0, c = 0, and p = 1**

We have an identity matrix; denoted as $\mathbf{I_{3\times 3}} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \in CJ4A$(3 x3, L (I) = 1

Identity matrices are always of class 4A and has its Libra value = 1.

$I_{n\times n} \in CJ4A$(3 x3, L(I) = 1)

For **a, b, c and p = 0;** in the general formwe have a null matrix denoted as **0.**

$\mathbf{0} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \in CJ4A$(3 x 3, L(**0**) = 0)**.**

We conclude that $\mathbf{0} \in Z_L$

2. STRUCTURAL ALGEBRAIC PROPERTIES OF CLASS 4A

In this section we introduce fundamental operations on the member matrices of class 4A

As we have defined CJ4A = {A | n x n, L(A) = p; p ∈ R } where A = (a_{ij}) for all i and j from N.

2.1 Three Basic Criteria

We have, on the members of class 4A, three fundamental criteria and the entire structure will gradually be mounted on it. We develop general notations for the members of class 4A.

Let **A, B, and C** be the member matrices of class **4A** and **α** be a real constant.

Let A ∈ $CJ4A$ (n x n, L (A) = P_1), B ∈ $CJ4A$ (n x n, L (B) = P_2), and C ∈ $CJ4A$ (n x n, L (C) = P_3) where P_1, P_2, and P_3 are their libra values; which are real constants.

Let **A = $(\mathbf{a_{ij}})$, B = $(\mathbf{b_{ij}})$, and C = $(\mathbf{c_{ij}})$ for all i and j** ∈ **N**

2.1(a) Equality

Two matrices **A = B** ↔ $\mathbf{a_{ij}} = \mathbf{b_{ij}}$ for all i and j ∈ N; in this case L(A) = L(B)

It is very important to note that equality of libra values does not necessarily imply equality of matrices.

2.1(b) Addition

Two matrices of same order of the class 4A can be added. The resultant matrix found on addition is also a matrix of the same class.

If A ∈ $CJ4A$ (n x n, L (A) = P_1), B ∈ $CJ4A$ (n x n, L (B) = P_2) then their addition denoted as A+ B = C (say) is a matrix derived as A + B = C =($c_{ij} = a_{ij} + b_{ij}$ | (n x n, L (A + B) = L(C) = $P_1 + P_2$**)**

for all i and j ∈ N. The important property is L(A) + L(B) = L(C)

2.1(c) Multiplication by a Scalar

Let α ∈ R, then for A= (a_{ij}) ∈ $CJ4A$ (n x n, L (A) = P_1) = We denote multiplication of A by a scalar α by **αA** and define it as **αA = $(\alpha\, \mathbf{a_{ij}})$ for all i, and j** ∈ **N**

This property helps us derive two important notion widely useful in matrix algebra.

For **α = -1,** we have -1A = -A [which is the additive inverse matrix of the matrix A]

and for **α = 0 , 0A = 0 =** Null matrix.

3. GROUP OF MATRICES

With the definition of the special class- i.e. class 4A and above mentioned three criteria (2.1) l we are now well equipped to develop more results.

Theorem 1

Prove that the set defined by **CJ4A = {A| n x n, L(A) = p; p ∈ *R* } where A = (a_{ij})** for all i and j from N is a commutative group.

Proof:

We consider the above set CJ4A = {A| n x n, L(A) = p; p ∈ *R* } where A = (a_{ij}) for all i and j from N.

We define operation addition (+) on the members of the set.

+: CJ4A = {A| n x n, L (A) = p; p∈ *R* } ⟶ CJ4A = {A| n x n, L (A) = p; p∈ *R* }

For A ∈ $CJ4A$ (n x n, L (A) = P_1), B ∈ $CJ4A$ (n x n, L (B) = P_2) then their addition denoted as A+ B = C (say) is a matrix derived as A + B = C =($c_{ij} = a_{ij} + b_{ij}$| (n x n, L (A + B) = L(C) = $P_1 + P_2$**)** A, and B ∈ CJ4A, A+ B ∈ CJ4A = { C| n x n, L(C) = $P_1 + P_2 = P_3$; P_3 ∈ *R* }

Addition of matrices on members of class 4A is a binary operation. In addition to this, we have one more property. L(C) = L (A) + L (B)

3.1. Associative Property for Addition

Let A = (a_{ij}) ∈ $CJ4A$ (n x n, L (A) = P_1), B= (b_{ij}) ∈ $CJ4A$ (n x n, L (B) = P_2),

and C = (c_{ij}) ∈ $CJ4A$ (n x n, L (C) = P_3) where P_1, P_2, and P_3 are their libra values; which are real constants and **a_{ij}, b_{ij}, and c_{ij}** are real numbers.

As associative property holds true on the set of real numbers we have

$$\mathbf{a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}}$$

i.e. For the three matrices of the same class -4A and of the same order we have

A + (B + C) = (A + B) + C An associative law for matrix addition holds true for the members of **Class 4A**

3.2 Existence of Identity Element

For A ∈ CJ4A, as established above, ∃ a null matrix, **0** ∈ CJ4A [Both A and **0** having the same order**]**

∍ **A + 0 = 0 + A = A.** This null matrix **(= 0)** is called an identity matrix for the binary operation + on the members of the set CJ4A.

3.3 Existence of Additive Inverse

For A ∈ CJ4A, as established above ∃ a matrix –A ∈ CJ4A ∋ **A + (-A) = 0 = - A + A**

The matrices A and –A are additive inverses of each other.

With these properties on hand along with the binary operation establish that (CJ4A, +) is a **group.**

3.4 Abelian Group

We now check the members of the group for commutative property for the binary operation '+'.

For A, B ∈ CJ4A, by definition, A + B = $(\mathbf{a}_{ij}) + (\mathbf{b}_{ij}) = (\mathbf{a}_{ij} + \mathbf{b}_{ij}) = (\mathbf{b}_{ij} + \mathbf{a}_{ij}) = (\mathbf{b}_{ij}) + (\mathbf{a}_{ij})$**= B +A**

[As $\mathbf{a}_{ij}$, and $\mathbf{b}_{ij}$ are real values satisfying commutative property for addition.]

In addition to this we note that L (A) + L (B) = L (B) + L (A)

This in turn implies that **(CJ4A, +)** is a **commutative group / an Abelian group.**

At this stage it is important to note that and on the parallel arguments it can be established that the class $\mathbf{Z_L}$ is also a commutative group under matrix addition.

4.0 SPECIAL *ILLUSTRATION* AND PROPERTIES

We cite here a special example of the above form

$$A = \begin{bmatrix} p-k & p+k & p+k \\ p+2k & p & p-k \\ p & p & p+k \end{bmatrix} \in CJ4A \text{ (3x3, L(A) = 3p +k) where p and k } \in \mathbf{R} \qquad \textbf{(4)}$$

We cite here some important properties of the above matrix.

(a) $|A| = -2k^2(3p+k)$; $|A| = 0$ only if k = 0

[k = 0 ⟹ the matrix A is a scalar matrix with all constant entries. ∴ $|A| = 0 \Longrightarrow 3p = 0$ and so

$p = 0$. this in turn ⟹ $L(A) = 3p + k = 0$. Again it implies that $A \in Z_L$]

(b) Eigen Values are 3p+k, k, and -2k

Corresponding Eigen Vectors are **(1, 1, 1)′ , (0, -1, 1)′ , and (-1, 1, 0)′**

[It is rather important to note here that we have already proved in our previous paper on the matrices of class 3A, that in any classified square matrix, the 'Libra Value' is one of the Eigen Value. With information and Eigen values properties associated with the entries of the matrix, the remaining Eigen values can be traced and in turn it helps us find Eigen vectors also.]

(c) Now we find the inverse matrix of the non-singular matrix of class 4A.

If **k** and **(3p+k) ≠0 then |A| ≠ 0 and hence**

$$\mathbf{A}^{-1} = \frac{1}{k(3p+k)} \begin{bmatrix} -p & \frac{p+k}{2} & \frac{p+k}{2} \\ 2p+k & \frac{p+k}{2} & -\frac{5p+k}{2} \\ -p & -p & 2p+k \end{bmatrix} \in CJ4A \text{(3x3, L(A) =1/(3p +k))}$$

The given matrix A and its inverse both are members of the same class—i.e. $CJ4A$(3x3)

Recalling the libra value property and a few calculations, we have Eigen values for the matrix $\mathbf{A^{-1}}$.

Eigen Values are **1/(3p+k) 1/k 1/-2k**

Corresponding eigen Vectors are **(1 1 1)′ (0 -1 1)′ (-1 1 0)′**

4.1 Commutative Property

The most important and long awaited classical property—commutative property for matrix multiplication is found in one of the typical subset of class 4A.

A typical subset of class4A [and there are some more also] is such that their member matrices obey this additional property on the top that they form an algebraic structure ‘ Ring’; the structure, at this point, is a commutative ring.

The general format is $A = \begin{bmatrix} p-k & p+k & p+k \\ p+2k & p & p-k \\ p & p & p+k \end{bmatrix}$

This structure of square matrices for real values of ‘p’ and ‘k’ forms an infinite sub-class of class 4A .

We denote this by **‘C1J4A’**

i.e. **C1J4A ⊂ $CJ4A$(3x3, L(A) = 3p +k)**

Theorem 2:

Statement: Matrices of class **C1J4A** satisfy commutative property for matrix multiplication. Also deduce the corresponding libra value condition.

Proof: Let us consider two member matrices A and B of the class **C1J4A**

Let $A = \begin{bmatrix} p1-k1 & p1+k1 & p1+k1 \\ p1+2k1 & p1 & p1-k1 \\ p1 & p1 & p1+k1 \end{bmatrix} \in CJ4A$(3x3, L(A) = 3p1 +k1) where p1 and k1∈ R

and

Let $B = \begin{bmatrix} p2-k2 & p2+k2 & p2+k2 \\ p2+2k2 & p2 & p2-k2 \\ p2 & p2 & p2+k2 \end{bmatrix} \in CJ4A$(3x3, L(B) = 3p2 +k2) where p2 and k2 ∈ R

Here both the matrices A and B are conformable for matrix multiplication and we take their product.

Taking product **AB and BA,**

We have AB =

$$\begin{bmatrix} 3p1p2+p1k2+k1p2+3k1k2 & 3p1p2+p1k2+k1p2-k1k2 & 3p1p2+p1k2+k1p2-k1k2 \\ 3p1p2+p1k2+k1p2-2k1k2 & 3p1p2+p1k2+k1p2+2k1k2 & 3p1p2+p1k2+k1p2+k1k2 \\ 3p1p2+p1k2+k1p2 & 3p1p2+p1k2+k1p2 & 3p1p2+p1k2+k1p2+k1k2 \end{bmatrix} \quad \mathbf{(5)}$$

The same result is found on taking **BA.**

$\therefore \boldsymbol{AB = BA}$ **(6)**

[This can be, in detail, checked in the annexure -1.]

Also it is note-worthy that for the libra value relation of the product matrices AB and BA, if conformable, then

L (AB) = L(A)L(B) =(3p1 + k1) (3p2 + k2) = 9p1p2+ 3(p1k2 +p2k1) +k1k2

= L (B) L (A) = (3p2 + k2) (3p1 + k1) =9p1p2+ 3(p1k2 +p2k1) +k1k2 = L (BA)

We conclude that **L (AB) = L (BA)**

[Comment: If two matrices are conformable for multiplication then AB is defined and

L (AB) = L(A) L(B). If both are of the same order then L (AB) = L (BA)

We note some important points on the libra values.

If L (AB) = L (A). L (B) = L (P) L (Q) =L(PQ)

*1 The product matrix AB may or may not be equal to the product matrix PQ if they exist.

***2** All L (A), L(B), L(P), and L(Q) may be different real values.

e.g. L(A) = 4, L(B) = 6, L(P) = 3, and L(Q) = 8 and both AB and PQ do not exist.]

4.2 Commutative Property – Symmetric and Anti-Symmetric Matrices

At this stage, we denote two special forms – Symmetric and Anti-symmetric matrices of class 4A.

Symmetric Form: There are many symmetric forms and anti-symmetric forms of square matrices of class 4A. They are denoted as **CJS4A (n x n, L(A)= 3p+k)** and **CJA 4A(n x n, L(a)= 3p+k)** where **p** and **k** are real constants.

E.G. A = $\begin{bmatrix} p+k & p+m & p-m \\ p+m & p+2m+k & p-3m \\ p-m & p-3m & p+4m+k \end{bmatrix} \in \boldsymbol{CJS\ 4A}$**(3 x 3, L(A) = 3p +k)** where p and k $\in R$ **(7)**

This is a symmetric matrix of class 4A

In fact, **CJS 4A (n x n, L (A) = 3p + k)** $\subset CJ4A$(n x n, L(A) = 3p +k)

E.G. For p = 3, k = 1, and m = --2

A = $\begin{bmatrix} 4 & 1 & 5 \\ 1 & 0 & 9 \\ 5 & 9 & -4 \end{bmatrix} \in CJS\ 4A$(3 x 3, L(A) = 10)

We cite here one of the general forms of anti-symmetric form of matrices of class 4A.

B = $\begin{bmatrix} p+k & p+m & p-m \\ p-m & p+k & p+m \\ p-m & p-m & p+k \end{bmatrix} \in CJA\ 4A$(3 x 3, L (B) = 3p +k) where p and k are real constants **(8)**

In fact, **CJA 4A (n x n, L (A) = 3p + k)** $\subset$ $\boldsymbol{CJ4A}$**(n x n, L (A) = 3p +k)**

E.G. For p = 3, k = 1, and m = - 2

B = $\begin{bmatrix} 4 & 1 & 5 \\ 5 & 4 & 1 \\ 5 & 5 & 4 \end{bmatrix} \in CJA\ 4A$(3 x 3, L(B) = 10)

Theorem 3

Statement: Matrices of these two classes viz. ***CJS* 4*A*(3 x 3, L (A) = 3p +k)** and

CJA 4A (n x n, L (A) = 3p + k) observe commutative property for ‘matrix multiplication ‘.

Proof

Part1

Let A ∈ $CJS\ 4A$ (n x n, L(A) = 3p1 + k1) and B ∈ $CJS\ 4A$(n x n, L(B) = 3p2 +k2) **(9)**

Let A = $\begin{bmatrix} p_1 + k_1 & p_1 + m_1 & p_1 - m_1 \\ p_1 + m_1 & p_1 + 2m_1 + k_1 & p_1 - 3m_1 \\ p_1 - m_1 & p_1 - 3m_1 & p_1 + 4m_1 + k_1 \end{bmatrix} \in$ ***CJS* 4*A*(3 x 3, L (A) = 3p_1 + k_1)**

and B = $\begin{bmatrix} p_2 + k_2 & p_2 + m_2 & p_2 - m_2 \\ p_2 + m_2 & p_2 + 2m_2 + k_2 & p_2 - 3m_2 \\ p_2 - m_2 & p_2 - 3m_2 & p_2 + 4m_2 + k_2 \end{bmatrix} \in$ ***CJS* 4*A*(3 x 3, L (A) = 3 p_2 + k_2)**

Both are square matrices and conformable for matrix multiplication.

We find both AB and BA; the result is as follows; where the result in the first bracket, carries three entries separated by comma sign are the three terms of the first row of the resultant matrix AB. In the same way the other entries can be considered.

In the same way we have considered the product BA and find the same result.

AB=

$$[[k1\,k2 + k1\,p2 + k2\,p1 + 2\,m1\,m2 + 3\,p1\,p2, k1\,m2 + k1\,p2 + k2\,m1 + k2\,p1 + 5\,m1\,m2 + 3\,p1\,p2, -k1\,m2 + k1\,p2 - k2\,m1 + k2\,p1 - 7\,m1\,m2 + 3\,p1\,p2],$$
$$[k1\,m2 + k1\,p2 + k2\,m1 + k2\,p1 + 5\,m1\,m2 + 3\,p1\,p2, k1\,k2 + 2\,k1\,m2 + k1\,p2 + 2\,k2\,m1 + k2\,p1 + 14\,m1\,m2 + 3\,p1\,p2, -3\,k1\,m2 + k1\,p2 - 3\,k2\,m1 + k2\,p1 - 19\,m1\,m2 + 3\,p1\,p2],$$
$$[-k1\,m2 + k1\,p2 - k2\,m1 + k2\,p1 - 7\,m1\,m2 + 3\,p1\,p2, -3\,k1\,m2 + k1\,p2 - 3\,k2\,m1 + k2\,p1 - 19\,m1\,m2 + 3\,p1\,p2, k1\,k2 + 4\,k1\,m2 + k1\,p2 + 4\,k2\,m1 + k2\,p1 + 26\,m1\,m2 + 3\,p1\,p2]]$$

= BA

[For satisfying detail refer to annexure-2.]

In addition to this, we have an important property related to their libra value’

L (AB) = L (BA) = L (A) L (B)

This proves result of **part-1.[**For symmetric matrices of class ***CJS* 4*A*(3 x 3, L (A) = 3p +k)**]

Part 2

In this section we work on the set of anti-symmetric matrices of class 4a. The class is

CJA 4A (n x n, L (A) = 3p + k). For the two matrices A ∈ $CJA\ 4A$(n x n, L (A) = 3p1 + k1) and B ∈ $CJA\ 4A$(n x n, L(B) = 3p2 +k2) **(10)**

we have to show that **AB = BA**

Let A = $\begin{bmatrix} p_1+k_1 & p_1+m_1 & p_1-m_1 \\ p_1-m_1 & p_1+k_1 & p_1+m_1 \\ p_1-m_1 & p_1-m_1 & p_1+k_1 \end{bmatrix} \in CJA\ 4A$(3 x 3, L(A) = $3p_1 + k_1$)

and B = $\begin{bmatrix} p_2+k_2 & p_2+m_2 & p_2-m_2 \\ p_2-m_2 & p_2+k_2 & p_2+m_2 \\ p_2-m_2 & p_2-m_2 & p_2+k_2 \end{bmatrix} \in CJA\ 4A$(3 x 3, L(B) = $3p_2 + k_2$)

Both A and B are anti-symmetric square matrices of class CJA4A (3 x 3, L(A1) = 3p + k)

Then, as we worked in **part 1**for symmetric matrices of class **CJS4A (3 x 3, L(A) = 3p + k), we follow the same routine for anti-symmetric matrices also.**

AB=

$$[[k1\,k2 + k1\,p2 + k2\,p1 + 2\,m1\,m2 + 3\,p1\,p2, k1\,m2 + k1\,p2 + k2\,m1 + k2\,p1 - m1\,m2 + 2\,m1\,p2 + 3\,p1\,p2, -k1\,m2 + k1\,p2 - k2\,m1 + k2\,p1 + m1\,m2 + 3\,p1\,p2], [k1\,m2 + k1\,p2 + k2\,m1 + k2\,p1 - m1\,m2 + 2\,m2\,p1 + 3\,p1\,p2, k1\,k2 + k1\,p2 + k2\,p1 + 2\,m1\,m2 + 2\,m1\,p2 + 2\,m2\,p1 + 3\,p1\,p2, k1\,m2 + k1\,p2 + k2\,m1 + k2\,p1 - m1\,m2 + 2\,m2\,p1 + 3\,p1\,p2], [-k1\,m2 + k1\,p2 - k2\,m1 + k2\,p1 + m1\,m2 + 3\,p1\,p2, k1\,m2 + k1\,p2 + k2\,m1 + k2\,p1 - m1\,m2 + 2\,m1\,p2 + 3\,p1\,p2, k1\,k2 + k1\,p2 + k2\,p1 + 2\,m1\,m2 + 3\,p1\,p2]]$$

= BA. We have the result **AB = BA** **(11)**

We find both AB and BA; the result is as above; where the result in the first bracket, carries three entries separated by comma sign are the three terms of the first row of the resultant matrix AB. In the same way the other entries can be considered.

In the same way we have considered the product BA and find the same result.

[For satisfying detail refer to annexure-3.]

In addition to this, we have an important property related to their libra value'

L (AB) = L (BA) = L (A) L (B) **(12)**

This proves result of **part-2.[**For anti-symmetric matrices of class ***CJA* 4*A*(3 x 3, L(A) = 3p +k)**]

5.0 LINEAR TRANSFORMATION AND ELEMENTARY OPERATIONS ON MATRICES OF CLASS- 4A

At this point we feel that before discussing some basic transformations defining concepts of all the classifications should be given at glance.

A Square Matrix for Which

(1) Sum of each column entries remains constant and same for all the columns fall in class1 denoted as CJ1(n x n, L(A) =p); where L(A) is called the Libra value which shows the constant value.

(2) Sum of each row entries remains constant and same for all the rows fall in class2 denoted as CJ2(n x n, L(A) =p); where L(A) is called the Libra value which shows the constant value.

(3) Sum of each column entries and row entries remains constant and same fall in class3 denoted as CJ3(n x n, L(A) =p); where L(A) is called the Libra value which shows the constant value.

(4) Sum of each column entries, row entries, and entries of non-leading diagonal remains constant and same fall in class 4A denoted as CJ4A(n x n, L(A) =p); where L(A) is called the Libra value which shows the constant value. This class possesses many infinite sub- classes such that the member matrices possess commutative property for multiplication. We have shown three of such classes.

(5) Sum of each column entries, row entries, and entries of leading diagonal remains constant and Same fall in class 4B denoted as CJ4B(n x n, L(A) =p); where L(A) is called the Libra value which shows the constant value. This class possesses many sub-classes and their member matrices follow commutative property for matrix multiplication.

(6) Sum of each column, row, leading and non-leading diagonal entries; i.e. all the properties inherent in both classes CJ4A, and CJ4B fall in this class denoted as CJ5 (NXN, L(A) = p)

Now we address certain properties for the square matrices of order 3 x3. All these properties can be extended to the matrices of order n x n.

5.1 LINEAR TRANSFORMATION

In this unit we discuss elementary row operations and a linear transformation.

5.1(a) We define a linear transformation **T** as follows.

$$\mathbf{T: CJ4A \rightarrow CJ4B} \tag{13}$$

We consider square matrices of order nxn but show the cases for matrices of order 3x3 and the same routine can be extended to a square matrix of any order.

For A $(=a_{ij}) \in$ CJ4A , $T(a_{ij}) = (a_{j(4-i)})$ for all i and j = 1 to 3.

then T(A) ∈ **CJ4B.** Let T (A) = B (say)

This transformation changes matrices of class 4A to the matrices of class 4B.

Illustration:

Let $A = \begin{bmatrix} 4 & 1 & 5 \\ 1 & 0 & 9 \\ 5 & 9 & -4 \end{bmatrix} \in CJ4A$(3 x 3, L(A) = 10)

We have $T(A) = B = \begin{bmatrix} 5 & 1 & 4 \\ 9 & 0 & 1 \\ -4 & 9 & 5 \end{bmatrix} \in CJ4B$(3 x 3, L(A) = 10)

Some important deductions:

***1** This transformation interchanges the first column with the third column and vice-versa.

***2** The middle column remains unchanged and hence in accordance with the above property, we conclude that L (A) = L (T (A)).

i.e. This linear transformation does not change the libra values on this transformation.

***3** T (T (a_{ij}) = T^2 (a_{ij}) ∈ **CJ4A** and so on for successive applications.

5.1(b) Elementary row /column transformation of the first and third row/column converts a given matrix of class4A to the one of class4B and vice-versa.

$R_1 \sim R_3$: CJ4A → CJ4B; and also **(14)**

$C_1 \sim C_3$: CJ4A → CJ4B

In the case of a matrix of order **4x4 of class 4A,** we need simply interchange R_1 with R_4 and

R_2 with R_3

$R_1 \sim R_4$, $R_2 \sim R_3$ } Both these operations at a time carried out on a matrix of order 4 x 4 of **class4A** will transform it in to a matrix of **class 4B**. **(15)**

Illustration: Let us consider a matrix A of order 4 x4.

$$A=\begin{bmatrix} 3 & -2 & 5 & 3 \\ 2 & 3 & -3 & 7 \\ -4 & 1 & 0 & 12 \\ 8 & 7 & 7 & -13 \end{bmatrix} \in CJ4A(4\text{x}4,\ L(A) = 9)$$

Applying the above mentioned transformation; we have the resultant matrix as follows.

$$A=\begin{bmatrix} 8 & 7 & 7 & -13 \\ -4 & 1 & 0 & 12 \\ 2 & 3 & -3 & 7 \\ 3 & -2 & 5 & 3 \end{bmatrix} \in CJ4B(4\text{x}4,\ L(A) = 9)$$

The important point is about preservation of libra value.

5.1(c) A very important one which lowers down the class; i.e. from class CJ4A it converts to the matrices of class 3 is depicted as follows. Elementary row /column transformation of the first and second row/column or the second and the third row/column converts a given matrix of **class4A** to the one of **class 3** and vice-versa.

$R_1 \sim R_2$: CJ4A → CJ3; equivalently C_1 to C_3 also. **(16)**

$R_2 \sim R_3$: CJ4A → CJ3; equivalently C_2 to C_3 also.

We add an important outcome that from matrix of class3, by this route, it is not possible to reach matrix of class4A. i.e. The transformation is not reversible.

CONCLUSIONS

In addition to our article on commutative matrices [Ref. 6] this article is one of the breakthrough in identifying the infinite classes of square matrices possessing commutative property briefly described at three different places as the topics advance as should be. For classical minded friends the relevant proof for each unit is given in the annexure. It also adds Libra value property associated with Eigen value and Eigen vector. The content leads to on establishing all the necessary algebraic property necessary for the algebraic structure of a 'Field'

Also the problem is an open ended one which can be attended by adding the same Libra value property on the entries of leading diagonal and also incorporating both these properties. Such symmetrical structures on the square matrices of order n x n ,it can be applicable to the area of organic chemistry in which symmetrical structure of C_6H_6 (benzene) plays in important role.

VISION

The next in this sequence we are about on completion of algebraic structure of square matrices of 4B.

We have identified many properties in these classes which are, we think, special attributes to these classes. .

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APPENDICES

Annexure: We, as mentioned during the proceedings of the core units of this paper, give the necessary algebraic work that will allow us to claim the commutative property in the cases of different sub-classes of the major class4A.

Annexture-1

In annexure -1, we have shown the resulting terms of each step of the product **AB and BA]We consider the matrices A and B given by relation (4) and both being conformable for matrix multiplication we continue to perform their product.**

$$A = \begin{bmatrix} p1-k1 & p1+k1 & p1+k1 \\ p1+2k1 & p1 & p1-k1 \\ p1 & p1 & p1+k1 \end{bmatrix} \text{ and } B = \begin{bmatrix} p2-k2 & p2+k2 & p2+k2 \\ p2+2k2 & p2 & p2-k2 \\ p2 & p2 & p2+k2 \end{bmatrix}$$

Their product denoted as AB is as follows.

AB = AB

$$[[(P1-K1)(P2-K2)+(P1+K1)(P2+2K2)+(P1+K1)P2,(P1-K1)(P2+K2)+2(P1+K1)P2,(P1-K1)(P2+K2)+(P1+K1)(P2-K2)+(P1+K1)(P2+K2)],$$
$$[(P1+2K1)(P2-K2)+P1(P2+2K2)+(P1-K1)P2,(P1+2K1)(P2+K2)+P1P2+(P1-K1)P2,(P1+2K1)(P2+K2)+P1(P2-K2)+(P1-K1)(P2+K2)],$$
$$[P1(P2-K2)+P1(P2+2K2)+(P1+K1)P2,P1(P2+K2)+P1P2+(P1+K1)P2,P1(P2+K2)+P1(P2-K2)+(P1+K1)(P2+K2)]]$$

$$=\begin{bmatrix} 3p1p2+p1k2+k1p2+3k1k2 & 3p1p2+p1k2+k1p2-k1k2 & 3p1p2+p1k2+k1p2-k1k2 \\ 3p1p2+p1k2+k1p2-2k1k2 & 3p1p2+p1k2+k1p2+2k1k2 & 3p1p2+p1k2+k1p2+k1k2 \\ 3p1p2+p1k2+k1p2 & 3p1p2+p1k2+k1p2 & 3p1p2+p1k2+k1p2+k1k2 \end{bmatrix} \quad \textbf{(12)}$$

BA =

$$[[(P1-K1)(P2-K2)+(P1+2K1)(P2+K2)+P1(P2+K2),(P1+K1)(P2-K2)+2P1(P2+K2),(P1-K1)(P2+K2)+(P1+K1)(P2-K2)+(P1+K1)(P2+K2)],$$
$$[(P2+2K2)(P1-K1)+P2(P1+2K1)+P1(P2-K2),(P1+K1)(P2+2K2)+P1P2+P1(P2-K2),(P1+K1)(P2+2K2)+(P1-K1)P2+(P1+K1)(P2-K2)],$$
$$[(P1-K1)P2+P2(P1+2K1)+P1(P2+K2),P1(P2+K2)+P1P2+(P1+K1)P2,(P1+K1)P2+(P1-K1)P2+(P1+K1)(P2+K2)]]$$

$$BA=\begin{bmatrix} 3p1p2+p1k2+k1p2+3k1k2 & 3p1p2+p1k2+k1p2-k1k2 & 3p1p2+p1k2+k1p2-k1k2 \\ 3p1p2+p1k2+k1p2-2k1k2 & 3p1p2+p1k2+k1p2+2k1k2 & 3p1p2+p1k2+k1p2+k1k2 \\ 3p1p2+p1k2+k1p2 & 3p1p2+p1k2+k1p2 & 3p1p2+p1k2+k1p2+k1k2 \end{bmatrix} \quad \textbf{(13)}$$

On comparing the results (12) and (13), we conclude that the class of matrices under

class 4-A preserves commutative property for multiplication of matrices.

i.e. in general AB = BA

This helps claim that AB = BA

Annexture-2

[In annexure -1, we have shown the resulting terms of each step of the product **A_1B_1 and B_1A_1.]**

In this annexure we consider two symmetric matrices of class 4-A. [form given by relation (11)]

Let A_1 = $\begin{bmatrix} \boldsymbol{p1+k1} & \boldsymbol{p1+m1} & \boldsymbol{p1-m1} \\ \boldsymbol{p1-m1} & \boldsymbol{p1+k1} & \boldsymbol{p1+m1} \\ \boldsymbol{p1+m1} & \boldsymbol{p1-m1} & \boldsymbol{p1+k1} \end{bmatrix} \in CJS(3, L(A) = 3p1+k1)$

and**B_1** = $\begin{bmatrix} \boldsymbol{p2+k2} & \boldsymbol{p2+m2} & \boldsymbol{p2-m2} \\ \boldsymbol{p2-m2} & \boldsymbol{p2+k2} & \boldsymbol{p2+m2} \\ \boldsymbol{p2+m2} & \boldsymbol{p2-m2} & \boldsymbol{p2+k2} \end{bmatrix} \in CJS(3, L(A) = 3p2+k1)$

These matrices are conformable for matrix multiplication

Their product AB and BA is given as follows. We have

$A_1B_1 =$

$$[[(P1 + K1)(P2 + K2) + (P1 + m1)(P2 - m2) + (P1 - m1)(P2 + m2), (P1 + K1)(P2 + m2) + (P1 + m1)(P2 + K2) + (P1 - m1)(P2 - m2), (P1 + K1)(P2 - m2) + (P + m1)(P2 + m2) + (P1 - m1)(P2 + K2)],$$
$$[(P1 + K1)(P2 - m2) + (P1 + m1)(P2 + m2) + (P1 - m1)(P2 + K2), (P1 + K1)(P2 + K2) + (P1 + m1)(P2 - m2) + (P1 - m1)(P2 + m2), (P1 + K1)(P2 + m2) + (P1 + m1)(P2 + K2) + (P1 - m1)(P2 - m2)],$$
$$[(P1 + K1)(P2 + m2) + (P1 + m1)(P2 + K2) + (P1 - m1)(P2 - m2), (P1 + K1)(P2 - m2) + (P1 + m1)(P2 + m2) + (P1 - m1)(P2 + K2), (P1 + K1)(P2 + K2) + (P1 + m1)(P2 - m2) + (P1 - m1)(P2 + m2)]]$$

$$A_1B_1 = \begin{bmatrix} 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + m1k2 + m1m2 & 3p1p2 + p1k2 + k1p2 - k1m2 - m1k2 + m1m2 \\ 3p1p2 + p1k2 + k1p2 - k1m2 - m1k2 + m1m2 & 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + k1p2 + k1m2 + m1k2 + m1m2 \\ 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 \end{bmatrix} \quad \textbf{(14)}$$

In the same way we find $B_1A_{1.}$

$B_1A_1 =$

$$[[(P1 + K1)(P2 + K2) + (P1 + m1)(P2 - m2) + (P1 - m1)(P2 + m2), (P1 + K1)(P2 + m2) + (P1 + m1)(P2 + K2) + (P1 - m1)(P2 - m2), (P1 + K1)(P2 - m2) + (P + m1)(P2 + m2) + (P1 - m1)(P2 + K2)],$$
$$[(P1 + K1)(P2 - m2) + (P1 + m1)(P2 + m2) + (P1 - m1)(P2 + K2), (P1 + K1)(P2 + K2) + (P1 + m1)(P2 - m2) + (P1 - m1)(P2 + m2), (P1 + K1)(P2 + m2) + (P1 + m1)(P2 + K2) + (P1 - m1)(P2 - m2)],$$
$$[(P1 + K1)(P2 + m2) + (P1 + m1)(P2 + K2) + (P1 - m1)(P2 - m2), (P1 + K1)(P2 - m2) + (P1 + m1)(P2 + m2) + (P1 - m1)(P2 + K2), (P1 + K1)(P2 + K2) + (P1 + m1)(P2 - m2) + (P1 - m1)(P2 + m2)]]$$

$$B_1A_1 = \begin{bmatrix} 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + m1k2 + m1m2 & 3p1p2 + p1k2 + k1p2 - k1m2 - m1k2 + m1m2 \\ 3p1p2 + p1k2 + k1p2 - k1m2 - m1k2 + m1m2 & 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + k1p2 + k1m2 + m1k2 + m1m2 \\ 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 & 3p1p2 + p1k2 + k1p2 + k1k2 - 2m1m2 \end{bmatrix} \quad \textbf{(15)}$$

On comparing the results (14) and (15), we conclude that the class of symmetric matrices under

class 4-A preserves commutative property for multiplication of matrices.

i.e. in general AB = BA